THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 23, 1901. Thirty-two persons were in attendance at the two sessions, including the following twenty-eight members of the Society:

Professor Anne L. Bosworth, Professor Joseph Bowden, Professor E. W. Brown, Dr. J. E. Clarke, Professor F. N. Cole, Professor T. S. Fiske, Mr. A. S. Gale, Mr. F. A. Giffin, Mr. Edwin Haviland, Jr., Dr. H. E. Hawkes, Mr. S. A. Joffe, Dr. Edward Kasner, Mr. C. J. Keyser, Professor Pomeroy Ladue, Dr. G. H. Ling, Dr. James Maclay, Dr. G. A. Miller, Mr. H. B. Mitchell, Professor E. H. Moore, Professor W. F. Osgood, Professor James Pierpont, Dr. M. B. Porter, Professor P. F. Smith, Professor Henry Taber, Professor J. H. Van Amringe, Professor E. B. Van Vleck, Miss E. C. Williams, Professor R. S. Woodward.

The President of the Society, Professor Eliakim Hastings Moore, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor John F. Downey, University of Minnesota, Minneapolis, Minn.; Professor Frederick C. Ferry, Williams College, Williamstown, Mass.; Mr. Henry T. Gerrans, Oxford University, Oxford, England; Mr. Edwin Haviland, Jr., New York, N. Y.; Professor Augustus E. H. Love, Oxford University, Oxford, England; Mr. V. R. Thyagarajaiyar, Bangalore, India. Two applications for membership were received. Action was taken by the Council toward making the Society's library accessible for the use of the members.

The following papers were read at this meeting:

(1) Dr. H. E. HAWKES: "Note on Hamilton's determination of irrational numbers."

(2) Professor E. B. VAN VLECK: "On the convergence of continued fractions with complex elements."

(3) Dr. M. B. PORTER: "On linear homogeneous finite difference equations, with applications to certain theorems of Sturm."

(4) Professor L. E. DICKSON: "Concerning real and complex continuous groups."

(5) Professor E. O. LOVETT: "An application of infinite groups to non-euclidean geometry."

(6) Professor E. O. LOVETT: "Contact transformations which change asymptotic lines into lines of curvature."

(7) Professor H. B. NEWSON: "Indirect circular transformations and mixed groups."

(8) Mr. W. B. FITE: "On metabelian groups that cannot be groups of cogredient isomorphisms."

(9) Dr. EDWARD KASNER: "On algebraic potential curves."

(10) Professor MAXIME BÔCHER: "Green's functions in space of one dimension."

(11) Dr. H. E. HAWKES: "Estimate of Benjamin Peirce's linear associative algebra."

(12) Dr. G. A. MILLER: "On holomorphisms and primitive roots."

(13) Dr. EDWARD KASNER: "Theorems on collinear lines in space."

(14) Mr. C. W. M. BLACK: "Decomposition of a form in n variables in an arbitrary domain with respect to a prime ideal modulus."

(15) Professor MAXIME BÔCHER: "An elementary proof of a theorem of Sturm."

(16) Dr. L. P. EISENHART: "Surfaces whose first and second fundamental forms are the second and first respectively of another surface."

(17) Dr. L. P. EISENHART: "Possible triply asymptotic systems of surfaces."

(18) Dr. H. F. STECKER: "On the determination of surfaces capable of conformal representation upon the plane in such a manner that geodetic lines are represented by algebraic curves."

(19) Professor MAXIME BÔCHER: "Non-oscillatory linear differential equations of the second order."

Mr. Fite was introduced by Dr. G. A. Miller, and Mr. Black by Professor James Pierpont. In the absence of the authors, Professor Bôcher's first paper was read by Professor W. F. Osgood, and the papers of Professor Dickson, Professor Lovett, Professor Newson, Dr. Eisenhart, Dr. Stecker, and the second and third papers of Professor Bôcher were read by title.

Professor Newson's paper appeared in the March number of the BULLETIN. The present number contains the first paper of Professor Bôcher, the first paper of Dr. Hawkes, and the second paper of Dr. Eisenhart. The papers of Professor Dickson and Dr. Miller, and Professor Bôcher's second paper will appear in later numbers. The paper of Dr. Stecker and the third paper of Professor Bôcher will

appear in the *Transactions*. Abstracts of the other papers are given below.

The paper of Professor E. B. Van Vleck, which will be published in the *Transactions*, gives first a résumé of the few existent theorems of a general character upon the convergence of continued fractions with complex elements. It then goes on to develop a set of equations which appears to be both new and fundamental for the study of continued fractions with complex elements. These equations are applied to the derivation of certain theorems, the first of which is as follows: If in a continued fraction

$$\frac{1}{a_1 + \imath\beta_1} + \frac{1}{a_2 + \imath\beta_2} + \frac{1}{a_3 + \imath\beta_3} + \cdots$$

the β-constituents with even subscripts have one sign and those with odd subscripts the opposite sign, while the α-constituents have a common sign, the continued fraction will converge if $\sum_{n=1}^{\infty} \left| a_n + \imath\beta_n \right|$ is divergent. On the other hand, if $\Sigma \left| a_n + \imath\beta_n \right|$ is convergent, the even convergents approach one limit and the odd convergents another. It is further shown that neither the numerator nor the denominator of any one of the convergents of the continued fraction can vanish. This also holds true for the convergents of any continued fraction in which either the a_n or the β_n meet the conditions of the first theorem.

Continued fractions are next considered in which a finite number of the α-constituents or of the β-constituents (but not of both) fail to fulfill the conditions imposed in the first theorem, and such continued fractions are shown to converge. Under certain restrictions, also, an infinite number of exceptional elements may be admitted. The criterion for convergence stated in the first theorem accordingly differs from criteria hitherto obtained in that it admits of certain irregularities in the continued fraction whose positions therein are entirely arbitrary.

In conclusion, the results of the paper are applied to algebraic continued fractions. Thus it appears that if each element λ_n of a continued fraction

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots$$

is a positive number or such a number multiplied into z, the zeros of the numerators and denominators of the convergents will all lie upon the negative half of the z-axis. If,

furthermore, no element with even (respectively with odd) subscript contains z and if $\sum |\lambda_n|$ is divergent, the continued fraction will converge over at least three-quarters of the plane; namely, over the half plane in which x is positive and in those sections in which x is negative and $|y| > |x|$. Within this region the continued fraction represents an analytic function and never vanishes. On the other hand, if $\sum |\lambda_n|$ is convergent, the even and odd convergents approach different limits, and these limits are likewise analytic and never vanish within the same region.

The theorems given by Sturm in his first important memoir in *Liouville's Journal*, Volume I., were first arrived at, so the author states at the end of the paper, as limiting cases of certain analogous theorems concerning linear difference equations of the second order. Sturm's work never having been published, Dr. Porter's paper aims at its restoration.

In the sixteenth volume of the *Acta Mathematica*, M. Zorawski followed a suggestion of the late Sophus Lie and determined the deformation invariants of surfaces in ordinary space by the method of the theory of continuous groups. It is the object of Professor Lovett's first paper to employ the same method to design the mechanism for the construction of the deformation invariants of varieties in a point space of any number of dimensions some power of whose lineal element is a homogeneous function of the differentials of the point coördinates of the space, and in particular to point out how the known and possible deformation invariants of surfaces in ordinary space and the theorems relative thereto may be extended immediately to surfaces in any quadratic space, euclidean or non-euclidean, of three dimensions. This particular result for varieties having three dimensions is especially interesting since it constitutes a contribution from the theory of groups to the geometry of all such varieties, and this in the face of the fact that the theory of finite groups is limited in its application to the construction of the geometries of three-dimensional manifoldnesses as is evidenced by the recent work of Bianchi (*Memoirs of the Italian Society of Sciences*, 3d series, vol. 11) and Cotton (thesis presented to the Faculty of Sciences of Paris), who have found all such varieties whose lineal elements admit of continuous groups.

The conclusion drawn follows from the fact that the deformation invariants of any particular class are determined

once for all for every variety of three dimensions by the integration of one and the same formal system of linear partial differential equations. To construct this system of linear equations it is necessary to effect an m-fold extension of the infinitesimal transformations of an infinite group. Such an extension can be proposed in an infinite number of ways. The latter problem has also been studied by Zorawski (*Proceedings of the Cracow Academy of Sciences*, 2d series, vol. 4), who extends the transformation relative to the derivatives of its own functions: and by Levi-Civita (*Transactions of the Venetian Academy of Sciences, Letters and Arts*, 7th series, vol. 5), who extends the transformation relative to the elements of any covariant and contravariant systems whatever. The present note extends the transformation in three different ways, namely relative to all partial derivatives of any function whose variation is known, with regard to all derivatives of any one variable considered as a function of the other point coördinates, and finally with reference to the derivatives of a system of any number of invariant functions. The corresponding systems of differential equations give generalizations of the so-called deformation invariants of Gauss, Beltrami, and Minding.

The geometry of contact transformations calls for the determination of those transformations which change a surface into another surface in such a manner that a remarkable family of curves on one surface is transformed into a remarkable family of curves on the other surface. Such, for example, are those establishing correspondences between asymptotic lines, lines of curvature, Darboux lines, geodesic circles, etc. Those transforming asymptotic lines into such are known to be either projective or dualistic. This result is generalized in a subsequent note. Dilatations and inversions are known to change lines of curvature into lines of curvature. These theorems relative to asymptotic lines and lines of curvature are to be found in Darboux's Theory of Surfaces, volume I. It is proposed in Professor Lovett's second paper to determine those transformations which change asymptotic lines into lines of curvature. One such transformation was found by Sophus Lie, namely his celebrated correspondence between straight lines and spheres, in his well known memoir in the fifth volume of the *Mathematische Annalen*.

It appears that the functions X, Y, Z, P, Q, capable of defining a contact transformation possessing the desired prop-

erty, must satisfy a certain system of equations of condition. The problem of determining the forms of the functions X, Y, Z, P, Q, from these general equations of condition offers serious difficulties. However, the means are at hand for the complete resolution of the problem, for Lie has remarked that all contact transformations which change asymptotic lines into lines of curvature also transform straight lines into spheres, and the latter transformations have been found by the author in a paper soon to be published. Accordingly, to find all contact transformations which change asymptotic lines into lines of curvature it is only necessary to subject the types of line sphere transformations already found to the equations of condition, a process simple but extremely laborious. It can be verified, however, without much difficulty that the functions X, Y, Z, P, Q determined by the sets of equations

$$\omega_1 = \sum_1^4 (a_{1j}X + ia_{1j}Y + a_{2j}Z + a_{3j})\,x_j = 0,$$

$$\omega_1 = \sum_1^4 (a_{2j}X - ia_{2j}Y - a_{1j}Z + a_{4j})\,x_j = 0,$$

$$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = 1,\ i = \sqrt{-1},$$

$$\omega_1^{\,x}/\omega_2^{\,x} = \omega_1^{\,y}/\omega_2^{\,y} = \omega_1^{\,X}/\omega_2^{\,X} = \omega_1^{\,Y}/\omega_2^{\,Y}$$

$$\omega_3 = \sum_1^4 a_{1j}X - ia_{1j}Y + a_{3j}Z + a_{3j})x_j = 0,$$

$$\omega_4 = \sum_1^4 (a_{2j}X + ia_{2j}Y - a_{1j}\,Y + a_{4j})x_j = 0,$$

$$x_1 = x,\quad x_2 = y,\quad x_3 = z,\quad x_4 = 1,\quad i = \sqrt{-1},$$

$$\omega_3^{\,x}/\omega_4^{\,x} = \omega_3^{\,y}/\omega_4^{\,y} = \omega_3^{\,X}/\omega_4^{\,X} = \omega_3^{\,Y}/\omega_4^{\,Y}$$

constitute two families of ∞^{15} transformations which change asymptotic lines of a surface into the lines of curvature of the transformed surface. The celebrated transformation of Lie is included in both families; we find it in the first by making

$$a_{21} = a_{42} = a_{33} = a_{14} = 1$$

and all the other constants zeros in the first set of equations, and also by putting

$$a_{11} = a_{32} = -\,a_{43} = -\,a_{24} = 1$$

and all the other coefficients zero in the second set of equations.

The above equations of condition can be simplified in several particulars. They reduce to fifteen in number if

$$B_1 = B_2 = B_3 = C_1 = C_2 = C_3 = D_1 = D_2 = D_3 = 0,$$

and these last equations are readily found to be compatible. They are satisfied by Lie's transformation; moreover for the latter we have also

$$\begin{vmatrix} X^s X^y \\ Y^s Y^y \end{vmatrix} = \begin{vmatrix} X_p X_q \\ Y_p Y_q \end{vmatrix} = 0, \quad A_2^2 - 4A_3A_1 = 0, \quad E_2^2 - 4E_3E_1 = 0.$$

It appears incidentally that the determinant

$$\begin{vmatrix} X^y X_p & X^y Y_p + X_p Y^y & Y^y Y_p \\ X^s X_p - X^y X_q & X^s Y_p + X_p Y^s - X^y Y_q - X_q Y^y & Y^s Y_p - Y^y Y_q \\ X^s X_q & X^s Y_q + X_q Y^s & Y^s Y_q \end{vmatrix}$$

is zero for any contact transformation.

Mr. Fite shows 1° that a metabelian group of odd order which has a set of generators such that the order of one of them is not a divisor of the least common multiple of the orders of all the others cannot be the group of cogredient isomorphisms of any group; 2° that if a metabelian group of order p^a, where p is a prime, has a cyclical commutator subgroup and a group of cogredient isomorphisms with more than two independent generators, it can not be the group of cogredient isomorphisms of any group.

The curves considered in Dr. Kasner's first paper present themselves in the theories of functions, of equations, and of the potential. They are defined analytically by the vanishing of a rational integral solution of Laplace's equation $\varphi_{xx} + \varphi_{yy} = 0$; or, what is equivalent, by the vanishing of the real part of a rational integral function of $x + iy$. The paper gives geometric definitions of these potential curves, and some of their metric and projective properties. A potential curve is apolar to the degenerate class conic composed of the circular points at infinity; all of its polar conics are rectangular hyperbolas. Any curve of the nth order passing through the n^2 foci of a curve of the nth class is a potential curve; conversely, any potential curve passes through the foci of an infinite number of systems of con-

focal curves of the nth class. Conjugate potential curves intersect orthogonally in a set of such foci. This last result is applied to the theory of equations.

In 1870 Benjamin Peirce enumerated all systems of hypercomplex numbers of certain types in less than seven units. His principles of classification are five in number and are shown to be identical or closely related to the principles of classification suggested by the theory of continuous groups and adopted by Scheffers (*Mathematische Annalen*, vol. 39 (1891), p. 253). In Dr. Hawkes's second paper various criticisms of Peirce's work by Cayley, Study, and others are noted and discussed.

Dr. Kasner's second paper is in abstract as follows: Consider any five collinear lines L_1, L_2, L_3, L_4, L_5, *i. e.*, five lines having a common tractor M_0. In addition to M_0, any four of these lines as L_2, L_3, L_4, L_5 have another tractor M_1. It is proved in the first place that the five lines M_1, M_2, M_3, M_4, M_5, thus derived, are themselves collinear. Denote their common tractor by L_0. The remainder of the paper is devoted to the study of the configuration composed of the twelve lines L and M and of the thirty points and thirty planes which they determine. The same configuration is determined by any five of the lines L, or any five of the lines M. The anharmonic ratios of the various points and planes are connected by simple relations; all are expressible in terms of four, the fundamental absolute invariants. Finally these relations are employed to prove that the derived set M_1, M_2, M_3, M_4, M_5 is correlative to the original set L_1, L_2, L_3, L_4, L_5.

In Mr. Black's paper it was shown first that all coefficients of any possible factor can be brought within the residual system of the modulus, and then the irreducible factors can be determined by a finite number of trials. In the second part of the paper, the uniqueness of decomposition is shown, first for a single variable, and then by induction for n variables, the main part of the discussion being directed to bringing the problem into a form such that reasoning similar to that used for the proof of the uniqueness of algebraic decomposition may be applied.

Dr. Eisenhart's first paper is in abstract as follows: In view of the fact that there is a unique surface corresponding to two quadratic differential forms, whose coefficients satisfy the Codazzi and Gauss equations, the question is asked

whether it is ever possible for the first and second fundamental forms of a surface to be the second and first respectively of a second surface. In answer it is found that there is a class of surfaces possessing this property ; that they are ruled surfaces with imaginary generatrices ; that the sphere of radius unity is the only real surface of the class ; and, furthermore, that the second surface differs from the first only by a translation in space.

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GREEN'S FUNCTIONS IN SPACE OF ONE DIMENSION.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society, February 23, 1901.)

I WISH to make a brief communication to the Society of some results which I have obtained, reserving proofs and further developments for another occasion.

By a Green's function is ordinarily understood a solution of Laplace's equation which vanishes on the boundary of a certain region, and within this region is discontinuous at only one point, where it becomes infinite like $1/r$ or $\log r$, according as we are dealing with Laplace's equation in three or in two dimensions. We may speak of this as a Green's function of the first kind, in distinction to the generalized Green's function for which more complicated conditions than the mere vanishing of the function are imposed on the boundary of the region.

This fundamental conception has been generalized by replacing Laplace's equation on the one hand by other homogeneous linear partial differential equations of the second order (cf. Encyklopädie, volume 2, p. 516); on the other hand by the ordinary differential equation $d^2y/dx^2 = 0$, which may be called Laplace's equation in space of one dimension (cf. Burkhardt, *Bulletin de la Société mathématique de France*, volume 22 (1894), p. 71). This suggests at once the possibility of considering in place of Laplace's equation the general ordinary homogeneous linear differential equation of the second order. I find that we can not only do this,

but can go further by considering at once linear differential equations of the nth order

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0 \qquad (n \geqq 2). \tag{1}$$

We will assume here, in order to avoid all complications, that $p_1, p_2, \cdots, p_n$ are real functions of the real variable x which are continuous at every point of the interval

$$a \leqq x \leqq b. \tag{J}$$

Let ξ be a point between a and b, and let k be a positive integer less than n $(1 \leqq k \leqq n-1)$.

We seek now a function $G(x)$ having the following properties:

(a) At every point of (J), except ξ, the function G is continuous, has continuous derivatives of the first n orders, and satisfies (1).

(b) At the point a, G and its first $n-1$ derivatives satisfy k linearly independent, but otherwise arbitrarily chosen, homogeneous linear relations.

(c) At the point b, G and its first $n-1$ derivatives satisfy $n-k$ linearly independent, but otherwise arbitrarily chosen, homogeneous linear relations.

(d) At ξ, G and its first $n-2$ derivatives are continuous.

(e) At ξ, G has a derivative D_+ of order $n-1$ taken in the positive direction, and a derivative D_- of order $n-1$ taken in the negative direction, and

$$D_+ - D_- = -1.$$

It turns out that in general there exists one and only one function satisfying these conditions. This function we call a Green's function, and denote it by $G(x, \xi)$. The effect of interchanging x and ξ is to change (1) into the adjoint equation (cf. Schlesinger's Handbuch, Vol. I., p. 53) and in general to alter the relations (b) and (c); otherwise conditions (a) — (e) remain true.

I will mention here only one application of the Green's function thus defined:

The solution of the non-homogeneous equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = p,$$

which satisfies at a and b the same conditions which are satisfied there by G, is given by the formula

$$y = -\int_a^b G(x, \xi)\, p(\xi)\, d\xi.$$

A special case deserves mention on account of its simplicity, viz., the case where conditions (b) reduced to

$$G(a, \xi) = G'(a, \xi) = \cdots = G^{(k-1)}(a, \xi) = 0$$

and conditions (c) to

$$G(b, \xi) = G'(b, \xi) = \cdots = G^{(n-k-1)}(b, \xi) = 0,$$

accents denoting differentiation with regard to x. One of the simplifications introduced by this specialization is that when x and ξ are now interchanged, the effect is to interchange a and b in conditions (b) and (c). It will be seen that for a given equation (1) and a given interval (J) there are $n-1$ Green's functions of this special sort, obtained by giving to k the values $1, 2, \cdots, n-1$. These functions might properly be called Green's functions of the first kind, since when $n = 2$ there is only one of them, namely the Green's function, which vanishes at both ends of (J).

GÖTTINGEN, GERMANY,
February 8, 1901.

ON A SYSTEM OF PLANE CURVES HAVING FACTORABLE PARALLELS.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, December 28, 1900.)

THE type of scrolls contained in a linear congruence, and having factorable asymptotic lines, gives rise to a class of plane curves whose parallels have a similar property.

It has been shown* that the spherical images of such scrolls have factorable lines of curvature and a method was given by means of which a scroll of this type could be transformed into an annular surface having only plane lines of curvature, the curve of each section breaking up into two factors. The planes of the system all belong to the same axial pencil, and the locus of centers of the generating spheres lies in the bisecting plane of the angle between the two plane directors.

* V. Snyder, "On a special form of annular surface." *Amer. Jour. of Math.*, vol. 23.

When the axis of this pencil of planes lies in the plane at infinity, the director planes become antiparallel, the generating spheres have a constant radius, and their envelope is a tubular surface. The procedure is as follows: take any two parallel lines

$$(q) \quad y = xt,\ z = 0; \qquad (p) \quad y + xt = 0,\ z = 0;$$

these lines go over into two parallel planes in sphere space. Take the conjugate of (p) with regard to the complex c, image of point spheres. This line (p') is $x = xz,\ t = 0$.

The image of this line (which does not cut (q)) is the same plane as that of p, but now regarded as generated by its second pencil of minimum lines, so that the direction of the normal is reversed. Any scroll having these two lines (p') and (q) for directrices will be defined by

$$\frac{y - xt}{z} = \lambda, \quad \frac{x - xz}{t} = \mu,$$

the relation between λ, μ being determined by the third director curve $f(\lambda, \mu) = 0$.

The spherical image of this scroll has for its equation

$$(1) \qquad \begin{vmatrix} \tau & 0 & \zeta & -(\xi + i\eta) \\ 0 & \tau & -(\xi - i\eta) & -\zeta \\ 0 & 1 & -\lambda & -x \\ 1 & 0 & -x & -\mu \end{vmatrix} = 0$$

or $$\left(\xi - \frac{(\mu + \lambda)\tau}{2}\right)^2 + \left(\eta - \frac{(\lambda - \mu)i\tau}{2}\right)^2 + \zeta^2 = x^2\tau^2,$$

i. e., it is the envelope of a sphere of constant radius x, whose center moves in the plane $\zeta = 0$, subject to the law $f(\lambda, \mu) = 0$. In particular, if $f(\lambda, \mu) = 0$ defines a scroll of the type mentioned, the lines of curvature of the second system belonging to this tube will be factorable. But it is at once evident that these curves are simply the parallels of the locus of centers.

Conversely, if such a tube is generated from a curve whose parallels are known to be factorable, the general annular surface derived from it and also the scroll, its line space image, will have the same property. Starting from this space, let $\varphi(\alpha, \beta) = 0$ be the equation of any plane curve; then

$$(\xi - (m\beta + \alpha))^2 + \eta^2 + \zeta^2 = \beta^2 + m^2\beta^2, \quad m = \frac{d\beta}{d\alpha},$$

is the equation of a sphere whose envelope is the surface of revolution formed by rotating $\varphi(\alpha, \beta) \equiv \varphi(\xi, \eta) = 0$ about the ξ-axis. This surface is an annular one contained in a linear congruence, its two director planes being $\zeta \pm i\eta = 0$. The image in line space of this sphere is

$$\beta\sqrt{1+m^2}z = x - (m\beta + a)t,$$
$$(m\beta + a)z = y - \beta\sqrt{1+m^2}t;$$

and the images of the directors are as follows:

$\zeta + i\eta = 0$ becomes $z + t = 0$, $x + y = 0$,
$\zeta - i\eta = 0$ " $z = t$, $x = y$.

The generators of the scroll having these two lines for directrices must be of the form

$$\frac{x+y}{z+t} = \lambda, \quad \frac{x-y}{z-t} = \lambda;$$

making them identical with the above form, we have

$$\frac{\mu+\lambda}{2}z = x + \frac{(\lambda-\mu)}{2}t, \quad \frac{(\mu-\lambda)}{2}z = y - \frac{(\lambda+\mu)}{2}t;$$

from which

$$\frac{\mu+\lambda}{2} = \beta\sqrt{1+m^2}, \quad \frac{\mu-\lambda}{2} = m\beta + a.$$

But when the directrices are transformed, the meaning of λ, μ remains the same and equation (1) shows what these directrices give when so transformed that the spherical image shall be a tubular surface. Hence: *the equations*

$$x = \beta\sqrt{1+\left(\frac{d\beta}{da}\right)^2}, \quad y = i\left(a + \beta\frac{d\beta}{da}\right)$$

$$\varphi(a, \beta) = 0, \quad \varphi(a, \beta) \neq \varphi(a, -\beta)$$

define a curve whose parallel curves break up into two distinct factors, corresponding to the positive and negative directions of the normal, i. e., toward or away from the center of curvature from the point on the curve.

It is not difficult to generate other surfaces, all of whose lines of curvature of both systems are plane, and those of one system factorable; any "surface moulure" having a factorable director curve is of this type,* but the above

* See Enneper, "Untersuchungen über die Flächen mit planen und sphärischen Krümmungslinien," *Göttinger Abhandl.*, vol. 23, cf. p. 45, eq. (68); and Bianchi, Differentialgeometrie, ch. 5, § 74.

surfaces are the only annular surfaces of the class. The second mantle of the surface of centers is the cylinder erected on the evolute of the locus of centers of the generating spheres. When this cylinder becomes a straight line the surface is a torus.

The factorability of the asymptotic lines of a scroll contained in a linear congruence depends entirely upon the configuration at the pinch points, all of which are situated on the linear directrices. When such a scroll is transformed into a tubular surface, these points go into the circular points of the plane containing the locus of centers: hence the configuration of the curve at the circular points determines whether its parallels be factorable or not.

In particular, when the equation of the scroll is of the form

$$\mu\varphi(\lambda) + f(\lambda) = 0,$$

the asymptotic lines are factorable when

$$\begin{vmatrix} \varphi & f \\ \varphi' & f' \end{vmatrix} \neq 0$$

is a perfect square (or a constant); but this is the condition that all of the pinch points are double, so that the n generators from each point of the multiple directrix are always the same, *i. e.*, z real, $n - z$ imaginary, z being constant.

Scrolls of this type exist for every value of n.

The corresponding condition for the curve is that $\alpha^2 + \beta^2$ should be equal to a square (or a constant), $\alpha^2 + \beta^2$ being the tangential equation of the circular points. For curves of class 2 or 4 this expresses that the foci of its dual curve must all be singular, but this is not true when the class is higher. Thus the dual of the general bicircular quartic has not factorable parallels, but those of the cartesian ovals, limaçon, cardioid, etc., have—the latter giving the cubic, $a(x^2 + y^2) = (a + 2x)^3$.

This equation $\varphi^2(\alpha, \beta, \gamma)(\alpha^2 + \beta^2) = f^2(\alpha, \beta, \gamma)$ is of more general application than to the $(n, 1)$ scrolls and seems necessary and sufficient in every case.*

* This equation is derived in a different way in Salmon's Higher plane curves, 3d edition, p. 103.

CORNELL UNIVERSITY,
October 30, 1900.

POSSIBLE TRIPLY ASYMPTOTIC SYSTEMS OF SURFACES.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, February 23, 1901.)

In my note, entitled "A demonstration of the impossibility of a triply asymptotic system of surfaces,"* which was read before the AMERICAN MATHEMATICAL SOCIETY at its December meeting, I failed to take account of an exception which presents itself in the discussion. It was found that in order that there exist a triply asymptotic system of surfaces, it is necessary and sufficient that the double system of asymptotic lines of each surface shall at the same time be geodesic lines on the surfaces. This double condition can be satisfied only when the asymptotic lines in both systems are rectilinear generators of the surface. The quadric surfaces are known to possess this peculiar property, and for the hyperboloid of one sheet and the hyperbolic paraboloid these generatrices are real. Hence instead of the general negation previously given we have the qualified one:

The only triple systems of surfaces cutting mutually in the real asymptotic lines of these surfaces are composed of properly associated families of hyperboloids of one sheet and hyperbolic paraboloids.

One such system can be gotten as follows:

As, in the previous note, we consider space referred to any system of curvilinear coördinates ρ_1, ρ_2, ρ_3 and let the cartesian coördinates x, y, z of a point with respect to fixed rectangular axes be given in terms of ρ_1, ρ_2, ρ_3 by the equations

$$(1)\quad x = f(\rho_1, \rho_2, \rho_3), \quad y = \varphi(\rho_1, \rho_2, \rho_3), \quad z = \psi(\rho_1, \rho_2, \rho_3).$$

We have remarked that the coefficients of the system

$$(2)\quad \begin{aligned} \frac{\partial^2\theta}{\partial\rho_1^2} &= a_{11}\frac{\partial\theta}{\partial\rho_1} + a_{12}\frac{\partial\theta}{\partial\rho_2} + a_{13}\frac{\partial\theta}{\partial\rho_3}, \\ \frac{\partial^2\theta}{\partial\rho_2^2} &= a_{21}\frac{\partial\theta}{\partial\rho_1} + a_{22}\frac{\partial\theta}{\partial\rho_2} + a_{23}\frac{\partial\theta}{\partial\rho_3}, \\ \frac{\partial^2\theta}{\partial\rho_3^2} &= a_{31}\frac{\partial\theta}{\partial\rho_1} + a_{32}\frac{\partial\theta}{\partial\rho_2} + a_{33}\frac{\partial\theta}{\partial\rho_3}, \end{aligned}$$

* BULLETIN, January, 1901.

can be so determined that it will admit x, y, z, as particular simultaneous solutions. And we have found that the necessary and sufficient condition that the families of surfaces

$$\rho_1 = \text{const.}, \quad \rho_2 = \text{const.}, \quad \rho_3 = \text{const.},$$

shall cut one another in their asymptotic lines is expressed by

$$a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0. \tag{3}$$

Consider now in particular the case where x, y, z are such that they satisfy the equations

$$\frac{\partial^2\theta}{\partial\rho_1^2} = 0, \quad \frac{\partial^2\theta}{\partial\rho_2^2} = 0, \quad \frac{\partial^2\theta}{\partial\rho_3^2} = 0. \tag{4}$$

In order that this system may be consistent with system (2), it is necessary that, either all the coefficients in the right hand members of the latter be zero, or that x, y, z, as functions of ρ_1, ρ_2, ρ_3, satisfy the condition

$$\frac{\partial(x, y, z)}{\partial(\rho_1, \rho_2, \rho_3)} = 0,$$

i. e., that there exist a relation between x, y, z. As this is impossible, we must have

$$a_{11} = a_{12} = \cdots = a_{33} = 0. \tag{5}$$

It is evident that the most general expressions which x, y, z can have under the conditions (4) are

$$\begin{aligned} x &= a_1\rho_1\rho_2 + b_1\rho_1\rho_3 + c_1\rho_2\rho_3 + d_1\rho_1 + e_1\rho_2 + f_1\rho_3 + g_1, \\ y &= a_2\rho_1\rho_2 + b_2\rho_1\rho_3 + c_2\rho_2\rho_3 + d_2\rho_1 + e_2\rho_2 + f_2\rho_3 + g_2, \\ z &= a_3\rho_1\rho_2 + b_3\rho_1\rho_3 + c_3\rho_2\rho_3 + d_3\rho_1 + e_3\rho_2 + f_3\rho_3 + g_3, \end{aligned} \tag{6}$$

where $a_1, b_1, \cdots, g_3$ are arbitrary constants. The cartesian coördinates of a point on one of the surfaces $\rho_3 = \text{const.}$ are given as the following expressions of the parameters ρ_1 and ρ_2, which evidently refer to the double system of lines in which the surface is cut by the two families of surfaces $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$,

$$\begin{aligned} x &= \alpha\rho_1\rho_2 + \beta\rho_1 + \gamma\rho_2 + \delta, \\ y &= \alpha_1\rho_1\rho_2 + \beta_1\rho_1 + \gamma_1\rho_2 + \delta_1, \\ z &= \alpha_2\rho_1\rho_2 + \beta_2\rho_1 + \gamma_2\rho_2 + \delta_2, \end{aligned} \tag{7}$$

where the expressions for $\alpha, \cdots, \delta_2$, as functions of $\rho_3, a_1, \cdots, g_3$, are readily found by comparing systems (7) and (6).

Eliminating ρ_1 and ρ_2 by means of Sylvester's dialytic method, we find the following equation of the second degree :

$$(a_1\beta_2)(a_1\gamma_2)x^2 + (a\beta_2)(a\gamma_2)y^2 + (a\beta_1)(a\gamma_1)z^2 - [(a_1\beta_2)(a\gamma_2) \tag{8}$$

$$+ (a_1\gamma_2)(a\beta_2)]xy + [(a_1\beta_2)(a\gamma_1) + (a_1\gamma_2)(a\beta_1)]xz -$$
$$[(a\beta_2)(a\gamma_1) + (a\gamma_2)(a\beta_1)]yz + Ax + By + Cz + D = 0,$$

where, for the sake of brevity, we have written

$$(a_i\beta_k) \equiv a_i\beta_k - a_k\beta_i,$$

and A, B, C, D are functions of $a, \cdots, \gamma_3$, whose form is of no consequence in the discussion except that they are finite for finite values of the latter.

Denoting by Δ the discriminant of the terms of second degree in (8), we find that

$$\Delta = 0. \tag{9}$$

In consequence of this and of the character of A, B, C, as remarked, the quadric whose equation is (8) has its center at an infinite distance. By giving to ρ_3 successive values in this equation, we get the equations of the form

$$f(x, y, z) = \text{const.},$$

defining the family of surfaces $\rho_3 = \text{const.}$ From (6) and (7) we see that the asymptotic lines on these surfaces are real when ρ_3 and $a_1, \cdots, g_3$ are real; hence these surfaces $\rho_3 = \text{const.}$ are of total negative curvature. Moreover, these asymptotic lines are distinct for general values of $a_1, \cdots, g_3$. Hence $\rho_3 = \text{const.}$ defines a family of hyperbolic paraboloids.

Since the formulæ (6) are perfectly symmetrical with respect to ρ_1, ρ_2, ρ_3, it follows that in a similar manner we can show that $\rho_1 = \text{const.}$ and $\rho_2 = \text{const.}$ define respective families of hyperbolic paraboloids. Hence the formulæ (6) serve to define a triply asymptotic system of hyperbolic paraboloids.

Consider the surface $\rho_3 = k_3$, where k_3 is a fixed constant. The cartesian coördinates of a point on this surface are given by (7), in which ρ_3 has been replaced by k_3. If in the expressions thus obtained we put ρ_2 equal to k_2, a fixed constant, we get the coördinates of a point whose locus is the asymptotic line in which the surfaces $\rho_3 = k_3$ and $\rho_2 = k_2$ intersect, asymptotic for both surfaces. These expressions may be written

$$x = l\rho_1 + p, \quad y = m\rho_1 + q, \quad z = n\rho_1 + r, \tag{10}$$

where $l, m, \cdots, r$ are constant along the line; their expressions are readily found from (6). The equations (10) show that the curve along which $\rho_3 = k_3$, and $\rho_2 = k_2$ intersect is a right line—a rectilinear generator for each surface—as we have seen from other considerations.

PRINCETON UNIVERSITY,
February, 1901.

NOTE ON HAMILTON'S DETERMINATION OF IRRATIONAL NUMBERS.

BY DR. H. E. HAWKES.

(Read before the American Mathematical Society, February 23, 1901.)

THE purpose of this note is to call attention to Hamilton's use of the partition (Schnitt) in his definition of certain irrational numbers.*

Though Hamilton's method of discussing the irrational number is open to serious criticism, it has some interest as affording an approximation to the rigorous theory afterward developed by Dedekind †, and also as presenting another example of the natural character of Dedekind's definition of the irrational number. ‡

For Hamilton time rather than space furnished the rudiment to which his intuition appealed for heuristic purposes and for his conception of unity. Starting with this conception he gives a careful exposition of the operations on positive and negative integers which do not carry one outside the system. A similar discussion of the rational number follows. From the analogy furnished by the laws of operations on integral and rational numbers, Hamilton writes expressions by which he defines the fundamental operations on ratios of any two intervals (steps) of time, and assumes that to any such ratio corresponds a number which he calls an "algebraic number." This he does, even though he has considered the existence of no numbers save integers and fractions In proceeding to develop his theory of irrational number Hamilton too often makes use of these "algebraic numbers," of whose existence and properties he has proved nothing. Throughout his work, however, fractions could be substituted for "algebraic numbers." and such a change would go far toward making his work rigorous In the following outline of Hamilton's work, it is assumed unless the

* "Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time," *Transactions of the Royal Irish Academy*, vol. 17 (1837), p. 293. This essay was read in 1833.

† Stetigkeit und irrationale Zahlen, 1872

‡ In commenting on Tannery's development of the partition (Introduction à la théorie des fonctions d'une variable, 1886) from a remark made by Bertrand (Traité d arithmétique, p. 203, 1849) Dedekind says, (Was sind und was sollen die Zahlen, 1893, p. xiii): "Diese Uebereinstimmung scheint mir ein erfreulicher Beweis dafür zu sein, dass meine Auffassung der Natur der Sache entspricht."

contrary is stated, that the numbers considered are "algebraic numbers."

Hamilton first shows as a corollary of a theorem on multiplication that for any numbers x' and x''

$$x'^2 \gtreqless x''^2 \text{ according as } x' \gtreqless x''.$$

He next assumes that between any two fixed numbers an infinite number of numbers must lie. Suppose A and B be two finite sets of numbers, arranged in order of magnitude, increasing and decreasing respectively, such that the greatest number in A is less than the least number in B. There must exist in general an infinite number of numbers less the least number in B and greater than the greatest number in A. Hamilton goes on to say that if the numbers in A and B are not given explicitly, but by some law, it may happen that no rational number can exist between A and B. But his intuition indicated that such sets could be used to determine non-rational numbers. This he elaborates in detail for a few particular cases, as for example the square root of a positive fraction. His work on this case is in outline as follows:

Let b denote a positive number. Suppose we have the sets A and B composed of an infinite number of rational elements

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots \quad \text{and} \quad \beta_1 > \beta_2 > \beta_3 \cdots,$$

respectively, where

$$\alpha_i^2 < b < \beta_j^2 \qquad (i, j = 1, \cdots).$$

Hamilton shows that these two sets A and B determine uniquely a quantity which he calls the irrational number a. He defines the product a^2 by the partition (A', B') where the elements of A' and B' are

$$\alpha_1^2 < \alpha_2^2 < \alpha_3^2 < \cdots \quad \text{and} \quad \beta_1^2 > \beta_2^2 > \beta_3^2 > \cdots,$$

respectively. He then shows that the number afforded by this partition is neither greater nor less than b. Thus he calls

$$a = \sqrt{b}.$$

Hamilton does not define addition or subtraction of partitions, nor does he order them. Neither does he show that every partition of real numbers defines a real number.

YALE UNIVERSITY,
February, 1901.

MUTH'S ELEMENTARTHEILER.

Theorie und Anwendung der Elementartheiler. Von Dr. P. MUTH. Leipzig, B. G. Teubner, 1899. xvi and 236 pages.

THE work under review covers an important subdivision of general invariant theory, a branch which deserves to be more widely known than it is. No doubt the cause of this neglect has been the lack of a text book, and Dr. Muth's monograph will help to remedy the defect.

The original problem which led to the series of investigations in this theory was that of the canonical reduction of two quadratic forms. In 1829 Cauchy published a paper on the secular inequalities of the planets,* in the course of which he showed that two quadratic forms can in general be reduced to sums of squares of the same variables. He also proved that the latent roots of the family† are all real, in the special case when one of the quadratic forms is positive for all real (non-zero) values of the variable, *i. e.*, is *definite*. Cauchy's results, though not perfectly general, cover most of the cases which occur in the first stages of geometry and dynamics. Jacobi (1834) found similar results by a somewhat different process.‡ Both Jacobi and Cauchy exclude the possibility of equal latent roots appearing in the problem.

The first systematic account of all possible types of two quadratic forms (allowing for equal latent roots) is to be found in Sylvester's paper (1851) on the contact of lines and surfaces of the second order.§ Here we meet with the idea of classification by means of *invariant factors.*|| Sylvester obtains 4 types of contact for conics, and 12 for quadrics; of course the algebraical possibility that the two forms differ only by a constant factor is trivial in a geometrical

* Exercices de mathématiques, vol. 4, p. 140 = Oeuvres (2d series), vol. 9 p. 174. Cauchy really discusses only the case when one of the original forms is already a sum of squares.

† If A, B are two given quadratic forms, the system of forms $uA + vB$ (u, v arbitrary parameters) will be called a *family* (as an equivalent for Kronecker's term *Schaar*) The *latent roots* are the values of the ratio $(-u : v)$ for which the determinant of the family vanishes; this determinant will be denoted by $|uA + vB|$.

‡ *Crelle*, vol. 12, p. 1 = Ges. Werke, vol. 3, p. 191.

§ *Phil. Magazine* (4th series), vol. 1, p. 119.

|| Weierstrass's *Elementartheiler*, and Sauvage's *élémentaire diviseur.* A definition is given in a later footnote on Weierstrass's work.

sense.* Unfortunately Sylvester did not prove that the invariant factors do constitute a *complete* set of invariants for two quadratic forms; nor did he explain how to effect the reduction of given forms to his standard types. †

In 1855 Cayley pointed out that Sylvester's ideas could be applied to the classification of homographies (*i. e.*, one-to-one transformations of space, or collineations). ‡ Here will be found a general formula for the number of possible collineations in space of any dimension n; and the calculation is given for $n = 1, 2, \cdots, 11$. Cayley gives symbols to represent each type by means of the indices of its invariant factors and enumerates the possibilities for $n = 1, 2, 3, 4$.§ In concluding the paper he says: "*Je reviendrai à cette théorie à une autre occasion*;" but apparently this promise was not fulfilled.

The next advance in the general theory was made in 1858 by Weierstrass, who gave a general method of reducing two quadratic forms to sums of the same squares.‖ He proved that if one of the forms be definite, the reduction is still possible, even if equalities exist among the latent roots. Using this result in connection with the dynamical problem of small oscillations about a position of equilibrium, Weierstrass showed that the stability is not destroyed by the presence of equal periods in the system; both Lagrange and Laplace supposed that equal periods would involve instability.¶ Weierstrass's theorem is commonly attributed

* In the algebraical problem we must further allow for the case of no contact, so that in all we have 6 and 14 as the numbers of the distinct types of families of quadratic forms ($n = 3$ or 4).

† To illustrate the simplification introduced by the use of invariant factors in geometry, the reader may compare the condition for double contact of two conics as found by Salmon (Conics, p. 346) with Sylvester's; which is simply that there should be two equal *linear* invariant factors of the determinant of the family.

‡ "Recherches sur les matricesdont les termes sont des fonctions linéaires d'une seule indéterminée." *Crelle*, vol. 50, p. 313 = Coll. Math. Papers, vol. 2, p. 216.

§ In his list for $n = 4$ the two symbols $\begin{matrix}31\\1\end{matrix}$, $\begin{matrix}21\\21\end{matrix}$ are omitted: Muth would write these [(21)1], [(11)(11)]. This is obviously an oversight on Cayley's part, as he states that 14 types are possible, but gives only 12. The indices of the invariant factors are the differences between Cayley's numbers.

‖ *Berl. Monatsberichte*, 1858, p. 207 = Ges. Werke, vol. 1, p. 233.

¶ The point of the theorem will probably be grasped better by considering a special (imaginary) example. Suppose we had a system with kinetic and potential energies, $T = x'y'$, $V = p^2xy + \frac{1}{2}x^2$, then the equations of motion are $x'' = -p^2x$, $y'' = -(p^2y + x)$, and corresponding to a term $A \sin pt$ in x we have one $\frac{At}{2p} \cos pt$ in y. Weierstrass proves that such a case is impossible in a real dynamical system.

by English writers* to Routh who rediscovered the theorem in his Adams prize essay (1877). Routh's elementary proof of the theorem requires to be supplemented by Weierstrass's algebraical theorem.

A beautiful and elementary proof of Weierstrass's theorem was given by Kronecker† in 1868, which appears to be less familiar than it ought to be.

Weierstrass‡ finally completed the theory of quadratic forms in 1868; and extended his theorems to the more general idea of bilinear forms. He proved that two families of forms are capable of transformation into each other if (and only if) the invariant factors§ of the two families are the same. The only case of exception to Weierstrass's results occurs when the determinants of the two families are identically zero; and this case was examined by Kronecker in a paper immediately following Weierstrass's (and quoted above). The complete statement of the conditions of equivalence in every case has been effected by Kronecker in a number of memoirs.‖

Invariant factors presented themselves from another point of view to H. J. S. Smith, who encountered them in connection with linear equations with integral coefficients.¶ He proves that many of the results known for the invariant factors of a family of forms hold for those of a matrix of integers.

We turn next to the historical development of Cayley's

* *E. g.*, Thomson and Tait, Natural Philosophy, vol. 1, § 343 *e* and *m*; where the reader will find some useful remarks on the physical side of the theorem. Routh was led to his theorem by the consideration of linear differential equations with constant coefficients; and in this connection obtained some other results due to Weierstrass, which were given originally in a communication to the Berlin Academy (in 1875), but were only published in vol. 2 of his works.

† *Berl. Monatsberichte*, 1868, p. 339 = Ges. Werke, vol. 1, p. 165.

‡ *Berl. Monatsberichte*, 1868, p. 310 = Ges. Werke, vol. 2, p. 19.

§ *Definition:* Suppose that the determinant of the family $(uA + vB)$ has a repeated factor $(au + bv)^p$; and further that *every* first minor is divisible by $(au + bv)^q$ but not by $(au + bv)^{q+1}$; *every* second minor by $(au + bv)^r$; and so on. Then the numbers $p - q$, $q - r$, ··· are the indices of the invariant factors to the base $(au + bv)$; they have the property $(p - q) \geqq (q - v) \geqq \cdots \geqq 1$, which was recognized by Cayley (l c., supra). The invariant factors can also be defined rationally, by means of highest common divisors (a remark due to Smith and Kronecker).

‖ For the results (which are rather long) the reader may consult § 8 of Dr. Muth's book; there is another investigation and a list of papers in the *Proc. Lond. Math. Soc.*, vol 32 (1900), p. 98.

¶ Math. Papers, vol. 1, p 367. and vol. 2. p. 623; some of Smith's results were published before the corresponding theorems for a family. For other references in this direction consult the Encyklopädie der Math. Wiss., vol. 1, pp. 582–597.

theory of matrices*; the foundations were given by him in 1855† and his first long memoir on the subject appeared in 1858‡; together with an application (in a consecutive paper) to Hermite's theory of the automorphic substitutions of a given quadratic form. Some years later (1867) Laguerre§ published an independent account of his calculus of "linear systems" which are virtually Cayley's matrices. But in 1853, Hamilton‖ had given certain results relating to a "linear and vector" function; which is essentially a matrix of order 3. For instance he proved that such a function satisfies a certain cubic equation with constant coefficients, and he found the reciprocal function. He therefore anticipated Cayley's theorem on matrices¶ (for the special case $n = 3$). Further, in 1867, he showed (Elements of quaternions) that the cubic may reduce to a quadratic (a special case of Frobenius's theorem referred to below). Probably this is the theory alluded to by Study (in the reference quoted above).

In a letter to *Nature* (vol. 44 (1891), p. 79), Professor J. W. Gibbs claims for Grassmann the first suggestion of the theory of matrices (Ausdehnungslehre, 1st ed., 1844). In this connection Whitehead (Universal algebra, Cambridge, 1898, vol. 1, p. 248) refers only to the second edition (1861) of Grassmann's work and implies that Cayley had anticipated Grassmann. I am unable to give any opinion on this point; nor do I know the extent of Grassmann's results.

In 1878 Frobenius** pointed out the important connection between bilinear forms and matrices (or linear substitutions). Being familiar with the results of Weierstrass and Kronecker on the equivalence of families of bilinear forms,†† he was naturally led to introduce the idea of invariant factors into the theory of matrices. This step has proved most fruitful in both theories; it enabled Frobenius to give the first general proof of Cayley's theorem, and to modify the theorem in the case when some of the latent roots of the matrix are

*Study in his report on complex units (Encyklopädie der Math. Wiss., vol. 1, p. 169) states that a theory had been hinted at by Hamilton. (See below.)

† *Crelle*, vol. 50. p. 282 = Coll. Math Papers, vol. 2, p. 185.

‡ *Lond. Phil. Trans.*, vol. 148, p. 17 = Coll. Math. Papers, vol. 2, p. 475.

§ *Jour. École Polyt.*, vol. 25, p. 215.

‖ Lectures on quaternions, pp. 559–569.

¶ That is: A matrix of order n satisfies an identical equation of order n. Apparently Cayley himself only verified the theorem up to $n = 3$; his proof is for the case $n = 2$ only.

** *Crelle*, vol. 84, p. 1.

†† It may be remarked that Frobenius gives a very convenient summary of these results in §6 of his paper.

equal.* Amongst other methods suggested in this paper is the use of the expansion of $(\lambda E - A)^{-1}$ to investigate properties of A, or rather of the latent roots of A; the expansion is made in powers of $(\lambda - a)$ (a being any latent root), and starts with a term in $(\lambda - a)^{-\alpha}$, $(\lambda - a)^{\alpha}$ being the first invariant factor of $|\lambda E - A|$, which belongs to a.†

Among other applications of this method we may mention Frobenius's definition ‡ of *any* function of A: provided that $f(A)$ can have a meaning, it is equal to the sum of the residues of $f(\lambda)$ $(\lambda E - A)^{-1}$ taken for all the latent roots of A. While speaking of this, I should call attention to what appears to be an oversight in the Clark University decennial volume (Worcester, Mass., 1899); in the report on the mathematical faculty, it is stated that Sylvester (in 1882) was the first to give an expression for the general power of a matrix. But according to Frobenius, Stickelberger (in 1881) gave the expression, and moreover for the case of repeated latent roots; while Sylvester assumes all the latent roots to be unequal.§ For the generalized function of a matrix Sylvester was the first to give a result ‖ (but with the same restriction on the latent roots); it is claimed in the Clark volume that the extension to the case of equal latent roots was made by Professor Henry Taber (in 1893,

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* By "latent roots" of a matrix A, we mean the roots of the determinantal equation $|\lambda E - A| = 0$, where E is the unit matrix. Probably the simplest proofs of the theorems are those of Frobenius (*Berl. Sitzungsberichte*, 1896, p. 604), though the same proof of Cayley's theorem had been given by Buchheim (*Mess. of Math.*, vol. 13 (1883), p. 62).

† To illustrate the method I collect six theorems on special matrices giving their authors and dates of publication. All the theorems can be proved most simply by Frobenius's method. (*L. R.* denotes latent roots; *I. F.*, invariant factors.)

A symmetrical	*L. R.* real (Cauchy, 1829).
	I. F. linear (Weierstrass, 1850).
A orthogonal	*L. R.* $e^{i\phi}$ (Brioschi, 1854).
	I. F. linear (Frobenius, 1878).
A alternate	*L. R.* imaginary (Weierstrass, 1879).
	I. F. linear (Weierstrass, 1879).

All these theorems were rediscovered by Professor Henry Taber (*Proc. Lond. Math. Soc.*, vol. 22 (1891), p. 449).

‡ *Berl. Sitzungsberichte*, 1896, p. 7.

§ After defining $f(A)$ as above, Frobenius (l. c., p. 11) says—"In dieser Weise hat Stickelberger in seiner akademischen Antrittsschrift: *Zur Th. d. lin. Diffgl.* (Leipzig, 1881) die allgemeine Potenz definirt und . . . benutzt. Eine weniger genaue Definition giebt Sylvester: *Sur les puissances et les racines des subst. lin.* (*Comptes rendus*, vol. 94 (1882), p. 55)." Professor Stickelberger informs me (in a private letter) that some copies of his paper were published at the end of 1880.

‖ *Johns Hopkins University Circulars*, vol. 3 (1882); of course the step from the generalized power to the general function is almost self-evident.

1894). But Buchheim* (in 1886) had already made the extension, and in a more concise form; both forms are, of course, contained in Frobenius's statement quoted above.

In referring to Frobenius we should remark that he has a claim to be a joint discoverer of a theorem, frequently attributed to Jordan,† to the effect that every *periodic* linear substitution can be reduced to the form

$$y_r' = e_r y_r$$

where e_r is a root of unity. This theorem has been recently taken up afresh by E. H. Moore and H. Maschke.‡ On the whole, the investigation of Frobenius seems the most direct and the clearest of all that I have seen; though, as it requires an elementary acquaintance with the theory of matrices, students of group-theory and differential equations may prefer an independent proof.

Of recent developments, we may refer to a recent paper by S. Kantor§ (1900) in which the theory of invariant factors of higher kinds (Stufen) is examined by the aid of geometry in space of n^2 dimensions. This appears to be an entirely fresh departure.

Dr. Muth's book is the first published account of the theories which we have sketched above;‖ our thanks are due to the author for having collected so many useful results into one convenient volume. On the other hand it may be questioned if readers beginning the subject will not do well to modify his arrangement. Thus §1, on general

* *Phil. Magazine*, 5th series, vol. 22, p. 173.

† On p. 16 of his paper, Frobenius shows that for periodic substitutions all the invariant-factors are linear and that the latent roots are roots of unity. On p. 21 it is stated that the reduced forms depend only on the invariant factors (using Weierstrass's results for bilinears), and the combination of these two facts leads directly to the theorem as given. Jordan's statement occurs later in the same volume of *Crelle* (p. 112); there can be little doubt as to their independence.

‡ *Math. Annalen*, vol. 50 (1898), pp. 215 and 220; cf. L. E. Dickson, "Report on progress in the theory of linear groups," BULLETIN, vol. 6 (1900), p. 13.

§ *Monatshefte für Math. und Phys.*, vol. 11, p 193; another way of applying n dimensional geometry to matrices has been given by Buchheim (*Proc. Lond. Math. Soc.*, vol. 16 (1885), p. 63) after Grassmann. We may consult also Whitehead's Universal algebra (vol. 1, p. 248).

‖ Two fairly complete accounts of the theory have appeared recently: Ed. Weyr. *Monatshefte für Math. und Phys.*, vol. 1 (1890), p. 163; Sauvage, *Ann. Ecole Norm. Sup.* (3d series), vol. 8 (1891), p. 285, and vol. 10 (1893), p. 9. Neither of these covers so much ground as Dr. Muth's work For sketches and references we may consult also F. Meyer's reports on general invariant theory; *Jahresbericht der Deutschen Math. Ver.*, vol. 1 (1890), p. 106 and Encyklopädie der Math. Wiss., vol. 1, p. 327.

properties of invariant factors, will probably be found one of the hardest in the book. In §2 will be found a good account of the theory of "multiplying" bilinear forms; the author follows Frobenius in preference to Cayley,* and introduces a matrix only as a picture (Bild) of the bilinear form. The theory of forms (or systems) with elements which are integers in various regions of rationality, occupies §§3–5 and 18. It may be pointed out that these methods can be applied† to deduce the principal theorems of §1.

In §§6–9 the theory of equivalence of families of forms is considered by the methods of Weierstrass and Kronecker; it seems to me that a somewhat easier introduction to the theory will be found in Darboux's paper on quadratic forms;‡ and in Stickelberger's extension of the same to bilinears;§ after which Muth's work will follow. It may be questioned if any advantage is obtained by the use of double suffixes in the reduced forms, at any rate in the tables (pp. 91, 116, 124, 133). The geometrical interpretation of the results on p. 124 by means of conics should be noted; this will be found in Sylvester's paper, already noted.

The remainder of the book is occupied chiefly with special applications; §10 contains Frobenius's method of reducing a given bilinear form by means of congruent substitutions;‖ § 11 gives a method of reducing a linear substitution to Jordan's canonical form, though it seems to me that the most practical method for reducing any substitution (whose latent roots are known) is that due to Jordan himself.¶

An application of the results of §11 is made in §16 to the theory of a system of linear differential equations with constant coefficients; this seems to be due to Weierstrass.

In §§12, 13, we have an account of linear substitutions which are automorphic for a given form; but it ought to be remarked that the first result in this direction is due to

* It is somewhat remarkable that Dr. Muth nowhere refers to Cayley—not even in his historical account of the subject.

† Cf. Hensel, *Crelle*, vol. 114 (1894), pp. 25 and 109. Two papers related to this point of view have just been published by Dr. Muth himself, *Crelle*, vol. 122 (1900), pp. 84 and 89.

‡ *Liouville's Jour. de Math.*, 2d series, vol. 19 (1874), p. 347.

§ *Crelle*, vol. 86 (1878), p. 20.

‖ This problem was attacked by Kronecker (in 1866) in connection with Weierstrass's general theta functions. Kronecker finally settled the problem in 1874, after some controversy with Jordan. The problem is of interest in certain dynamical questions, as well as in connection with theta functions.

¶ Cours d'Analyse, vol. 3, Art. 143; a reproduction is given in Craig's book on Differential equations. A series of papers on this subject will be found in vols. 30-32 of the *Lond. Math. Soc. Proc.* (1899–1900).

Hermite,* whose conclusions were translated into the language of matrices by Cayley. No references are given to recent researches on the so-called "singular" cases of automorphic substitutions.† It is perhaps worth while to call attention to a result of Loewy's ‡ which shows that, by using a Hermite form (with complex coefficients) in place of the real symmetrical and alternate forms hitherto considered, we avoid all consideration of the singular cases. Loewy's paper contains many other valuable results and will repay careful study.

In §§14, 15, Dr. Muth obtains certain results on linear invariant factors; here we should note Klein's results in his inaugural dissertation (1868)§, which have been further extended by Loewy.|| §17 contains an elaborate account of all possible collineations in space of n dimensions, which is afterwards applied to spaces of two and three dimensions. As stated before, the first attempt to classify such transformations is due to Cayley, who did not, however, include the possibility of "singular" collineations; as an illustration of what is meant by such collineations we may take the perspective of ordinary drawing, in which a point of the picture represents all the points on a certain line through the eye. Another form of classification has been given by Whitehead.¶

Apparently it is usual to conclude a review with a list of misprints. Of those which I have noticed, not many need delay the reader. In several places the short vertical lines

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* *Crelle*, vol. 47 (1854), p. 309 ($n=3$) and *Camb. and Dublin Math. Jour.*, vol. 9 (1854), p. 63 ($n=4$). In connection with Muth's table on p. 172 we may refer to Jordan, *Liouville's Jour. de Math.* (4th series), vol. 4 (1888), p. 349.

† See § 11 of Frobenius's paper; it had been considered previously (in the special case $n=3$) by Bachmann, Tannery and Hermite, without the aid of matrices. More recent investigations are those of Loewy in *Math. Annalen*, vol. 48 (1897), p. 97 and vol. 49, p. 448; of Taber, *Proc. Lond. Math. Soc.*, vol. 24 (1893), p. 290, and vol. 26 (1895), p. 364; also in *Math. Annalen*, vol. 46 (1895), p. 561, and a number of papers in the BULLETIN. In connection with Taber's work we should notice a paper by Rettger (*Amer. Jour. of Math.*, vol. 22 ('900), p. 62, which gives similar results from the point of view of general continuous group theory. (See also the Clark volume, already quoted.)

‡ *Nova Acta Leopoldina*, vol. 71 (1898), p. 379 = *Math. Annalen*, vol. 50, p. 569. Cf. *Göttinger Nachrichten*. 1900, p. 298.

§ Reprinted in *Math. Annalen*, vol. 23; the theorem alluded to is given on pp. 561, 562 and is, from the point of view of Klein's dissertation, a subsidiary result. In the reprint Klein remarks that the theorem seems not to have been sufficiently considered.

|| *Crelle*, vol. 122 (1900), p. 53.

¶ Universal algebra, vol. 1, p. 316.

(used to indicate determinants) have been misplaced. In the lists of reduced forms some small errors have caught my eye;* on p. 142 (footnote) in reference to Frobenius's paper in *Crelle*, vol. 86, we should read p. 146 for p. 20. Near the foot of p. 166, $(S+T)^{-1}\ (S+T)$ should be $(S+T)^{-1}\ (S-T)$; and in some places there are slight errors in the titles of papers quoted.

In conclusion I may say that Dr. Muth's book is of great interest and very useful in extending one's knowledge of certain branches of the subject. I hope that it may induce other readers to take up this part of invariant theory, which is important on account of its applications as well as for its intrinsic interest.

T. J. I'A. BROMWICH.

ST. JOHN'S COLLEGE, CAMBRIDGE,
January 1, 1901.

SHORTER NOTICES.

Kurzgefasste Vorlesungen über verschiedene Gebiete der höheren Mathematik, mit Berücksichtigung der Anwendungen. Von DR. ROBERT FRICKE. Large 8vo. Pp. ix + 520. Leipzig, B. G. Teubner, 1900.

Two objects are sought in Dr. Fricke's timely book: first, to supply a defect in German mathematical literature, a handbook for students who have mastered the elements of analysis and are not yet qualified to read profitably the highly specialized treatises; second, to smooth the way for technical students who discover a taste for the more abstract branches of mathematics. The present volume is confined to analysis and theory of functions, a second is announced as in preparation, to treat of advanced portions of algebra and geometry.

The reader is presumed to have a pretty thorough acquaintance with integral calculus, though not with the calculus of imaginaries. Fourier's series are first introduced, with applications to vibrating strings and to diffusion of heat. A short chapter is given to spherical and cylindrical harmonics, with tables for the functions $P_1(\mu)$, $P_2(\mu)$, $-P_6(\mu)$ according to Byerly, and of $J_0(\vartheta)$ and $J_1(\vartheta)$

* The second form in each of the following needs correction: p. 91 (c. 3); p. 117 (5) and (8). The last of these has an x instead of a y; in each of the other two a suffix has been misprinted.

according to Lommel. The remaining chapters are based upon the third. Functions of a Complex Variable, and it is in this narrow space of 100 pages that the skill of the author in concise and transparent exposition is most strikingly evident.

In attractive, inductive sequence, mainly in geometric style, are presented linear functions and inversion with respect to a circle, stereographic projection, typical rational functions, exponential and hyperbolic functions, then the general concept of analytic functions of the complex variable. The style is that of the lecturer rather than that of the critic, and the requirements of rigor are met by making definitions and hypotheses amply inclusive. At once the motion of an incompressible fluid is taken up briefly, and the question of a function whose integral $\int f(x, y)ds$ from point to point of a domain shall be independent of the path s. Thus are brought in the potential equation, Cauchy's integral, and Green's theorem, while the notion of analytic function is still in the formative stage in the student's mind. Power series, convergence, irregular points, entire functions developed in products, and the gamma function are rapidly sketched, the chapter closing with a well illustrated and suggestive paragraph on Riemann surfaces and the problems arising from them.

Elliptic functions are treated first by the Weierstrassian method, then conversely after Lagrange and Jacobi, as based upon integrals and theta functions. Here, as earlier for spherical harmonics, tables are given: for the elliptic integrals of the first and second kinds and their periods, and for $\log q$ as a function of k. These tables are from Lévy's Précis élémentaire etc. Numerous applications of elliptic functions fill 60 pages, among which the short paragraph on confocal quadric surfaces and elliptic coördinates is the one that will be most useful to teachers.

Two concluding chapters are devoted to ordinary and partial differential equations in complex variables, the method and applications being principally drawn from Fuchs. Klein, Schwarz, and Jacobi. These chapters, though condensed, are not crowded, but are throughout readable. Most teachers to whom the purpose indicated by Dr. Fricke appeals as eminently desirable will find this book full of helpful suggestions, and will moreover find their own interest in the successive topics vigorously stimulated by the occasional reading of a chapter. The second volume will be awaited with not a little of curiosity.

Two extracts from the preface are deserving of considera-

tion in this country at least. "— Ich darf wohl auf Zustimmung hoffen, wenn ich die Meinung äussere, dass zu einer tiefer gehenden Verständniss der Mechanik sowohl in ihren Grundproblemen als in ihren Einzelausführungen verschiedene mathematische Disciplinen erforderlich sind, die zur Zeit leider eine regelmässige Pflege an den technischen Hochschulen nicht finden. * * * In Amerika und England sehen wir vielfach ein Unterrichtssystem in Kraft, bei welchem mit relativ geringfügigen mathematischen Hilfsmitteln höchst ausgedehnte Gebiete der Anwendungen berührt werden. Der richtige Weg liegt auch hier gewiss in der Mitte. Um ihn mit Erfolg zu betreten, wird die Mathematik mit den technischen Wissenschaften unter gegenseitiger bereitwilliger Förderung Hand in Hand gehen müssen."

HENRY S. WHITE.

Ebene Geometrie der Lage. Von PROFESSOR DR. R. BÖGER. Leipzig, G. J. Göschen'sche Verlagsbuchhandlung, 1900. (Sammlung Schubert, VII.) 8vo. Pp. x + 289. Price 5 marks.

A NEW text book on the geometry of position is prepared by Dr. Böger, who is Professor at the Real-gymnasium in Hamburg. The book has some novelties, and distinctive merits. In the first place it is well arranged for reference. Section title and number and paragraph number stand at the head of alternate pages. Paragraph numbers appear black in the text, and are set also in smaller type in the outer margin. Definition, theorem, and corollary are distinctly marked as such. Thus the frequent references back and ahead are consulted with the least possible labor, a rapid review is easy, and the student is never left in doubt as to purpose or connection of a sentence. The table of contents is full and well divided, but there is no index. This lack however will not greatly hinder the student, for at every critical point a line of back references is struck which leads to every desired explanation.

As its title indicates, the work is confined to the plane. Beginning with the perspective relation, it concludes with the construction of a polar relation out of any five real or imaginary elementary data (pairs of conjugate points). Steiner and Cremona defined projective ranges to be any two members of a sequence, each member of which is in perspective with the next preceding and the next succeeding. This definition Böger adopts, and so proves as a theorem

the invariance of harmonic sets under projective transformation. Von Staudt and Reye chose the reverse order, but I think it probable that most teachers will agree with Böger as to the easier order for students. From two projective pencils or ranges a conic is generated; then follow the simpler properties; afterward the theory of involutions on a range or pencil, then upon a conic; and so are reached the relation of pole and polar and the usual metric properties of diameters and foci.

Here the second part of the work begins, wherein not the conic itself, but the polarity which it mediates, is the principal object. Whatever determines the one determines the other, and these determining elements may just as well be pairs of conjugate points as exclusively self-conjugate points. The problems of construction thus generalized are here solved by the aid of concepts that will be new to many readers: adjoint, conjoint, component, and resultant involutions. On a conic two involutions are *resultant*, for example, when their centers are conjugate with respect to the conic; and if two involutions are considered, both are *component* to their common resultant. The terms seem worthy of general adoption. A brief treatment of sheaves of polarities and of cubic involutions is added, breaking off where the next topic would be polarities of the third order.

Most striking is perhaps the absence of the word imaginary. From the preface we extract the author's reason for the omission. "Von Staudt's theory of imaginary elements, founded upon a normal Wurf, I have replaced by a theory based upon a perfectly arbitrary Wurf. This theory renders it needless to distinguish between real cases and imaginary, because it employs only such proofs as are valid for all cases alike. If we retain the involution that is determined by a Wurf, and base our proofs always on this involution, never on the presence or absence of its two coincidence points, then there will never be need to introduce imaginary elements. Further, the word imaginary is not only needless, it is positively a detriment; for since it has no corresponding image (Vorstellung), its effect can be nothing but confusion. Therefore the word imaginary ought to be outlawed from the geometry of position." This reasoning and a similar plea in the closing paragraph may not convince the reader, but they are well worth study to any one who is prone to mix algebraic short cuts with geometric reasonings.

HENRY S. WHITE.

NOTES.

At the regular meeting of the London mathematical society held on February 14, 1901, the following papers were read: "The distribution of velocity and the equations of the stream lines due to the motion of an ellipsoid in fluid frictionless and viscous," by Mr. T. Stuart; "On factorizable twin binomials," by Lieut. Col. A. J. Cunningham; "Concerning the abelian and related linear groups," by Professor L. E. Dickson; "A geometrical theory of differential equations of the first and second orders," by Mr. R. W. Hudson; "Brocardal properties of some associated triangles," by Mr. R. Tucker; "A note on stability, with a hydrodynamical application," by Mr. T. J. I'a. Bromwich; "Remarks on notation in Lie's theory of continuous groups, and on Schur's determination of a continuous group of given structure, with remarks on Mr. J. E. Campbell's paper (read at the January meeting)," by Mr. H. F. Baker; "Note on curves similar and parallel to one another," by Mr. D. B. Mair.

The publication of the works of Weierstrass, which the Prussian academy of sciences has undertaken, has been unexpectedly delayed, and little progress has been made during the past year. About three-quarters of volume IV. (Abelian functions) has now been printed.

The first number (1901) of the *Annuaire des mathématiciens*, published by Messrs. Carré and Naud, of Paris, is in press, and will soon appear. It will contain about 7,000 names with addresses, a list of the principal mathematical societies, and a list of mathematical periodicals. Besides this, the volume will contain certain scientific notices; among the authors of these notices will probably be Professors P. Appell, Gino Loria, D. Hilbert, F. Klein, C. Méray, J. Petersen, and P. H. Schoute. The preface will be by Dr. C A. Laisant, editor of *L'Enseignement mathématique*, to whom the publication of the volume is due.

The Prussian academy of sciences has received the sum of 100.000 marks from the city of Berlin. The money was given at the celebration of the two hundredth anniversary of the academy, and the interest on it is to be applied at intervals of four years to an important investigation in the field of the natural sciences. The first award is to be made in 1904.

CAMBRIDGE UNIVERSITY.—The Adams prize for 1901 has been awarded to H. M. MACDONALD, fellow of Clare, for an essay on "Electric waves." The subject for the Adams prize in 1903 is: "The bearing on mathematical physics of recent progress in the theory of the representation of discontinuous quantity by series, with special consideration of the logical limitations of the processes involved." The prize is open to the competition of all who have received a degree from the university. The successful candidate will receive about £225. Essays are to be sent to the vice-chancellor by December 16, 1902.

THE Naples academy of mathematical and physical science has awarded its mathematical prize of 1,000 lire for 1899 to Dr. G. TORELLI, professor at Palermo. The subject was the totality of prime numbers. The theme for the next award is the theory of invariants of the ternary biquadratic, considered preferably in relation to the condition for splitting into lower forms. The essays may be written in Italian, French, or Latin, and must be sent in, distinguished by a motto, before March 31, 1902.

THE royal academy of sciences of Turin reannounces the conditions made public January 1st, 1899, for its Bressa prize. The competition is open to the world and closes December 31st, 1902. The prize will be conferred on the discoverer of the most useful invention or the author of the most celebrated work in the mathematical and physical sciences, not excluding geology, history, geography and statistics. The value of the prize is nine thousand six hundred francs.

THE Madrid academy of sciences calls for a historical memoir on the Spanish mathematicians of the sixteenth century for its next annual prize in the mathematical sciences. Biographies and exhaustive accounts of works both published and unpublished are demanded.

UNIVERSITY OF OXFORD. The following announcements are those for the Hilary term 1901 in the mathematical sciences, the courses consisting in general of two lectures per week :—By Professor W. ESSON : Synthetic geometry of conics ; Synthetic geometry of cubics.—By Professor E. B. ELLIOTT : Elements of elliptic functions ; Supplementary lectures on quantics.—By Professor H. H. TURNER : Elementary mathematical astronomy.—By Professor A. E. H. LOVE : Dynamics ; Fourier's series and its physical applications.—By Mr. A. L. DIXON : Calculus of finite dif-

ferences.—By Mr. J. E. CAMPBELL: Algebra of quantics. —By Mr. P. J. KIRBY: Solid geometry.—By Mr. C. H. THOMPSON: Dynamics of a particle.—By Mr. E. H. HAYES: Geometrical optics.—By Mr. C. E. HASELFOOT: Physical optics.—By Mr. H. T. GERRANS: Hydrodynamics. By Mr. J. W. RUSSELL: Pure geometry.—By Mr. C. LEUDESDORF: Geometry (Maxima and minima, inversion, etc.).

UNIVERSITY OF PARIS. The following courses in the mathematical sciences are announced for the second semester at the faculty of sciences, the semester opening March 1st, 1901, and each course consisting of two lectures per week :—By Professor É. PICARD: Algebraic functions and the transcendents associated with them. The introductory lecture to the course is devoted to the scientific work of Hermite; this lecture will appear in the current volume of the *Annales de l'École Normale*.—By Professor E. GOURSAT: Differential equations.—By Professor P. APPELL: Continuation of the course of the first semester in mechanics.—By Professor J. BOUSSINESQ: Continuation of the course of the preceding semester in mathematical physics.—By Professor G. KOENIGS: Study of machines. —By M. C. BOURLET, replacing Professor L. RAFFY: Differential equations and their applications to mechanics and physics.—By Professor J. HADAMARD and M. M. SERVANT: Conferences on infinitesimal calculus.—By Professor P. PUISEUX: Conferences on mechanics and astronomy.—By Professor J. HADAMARD and MM. H. ANDOYER and E. BLUTEL: Conferences on the subjects for the agrégation.

THE several German universities below offer during the summer semester 1901 courses in mathematics as follows:

UNIVERSITY OF BRESLAU.—By Professor J. ROSANES: Plane analytic geometry, four hours; Theory of determinants, two hours; Seminar, one hour.—By Professor R. STURM: Theory of geometrical relationship, part one, three hours; Higher algebraic curves, three hours; Seminar on descriptive geometry, two hours.—By Professor J. FRANZ: The problem of three bodies, two hours; Calculus of finite differences, with applications to special perturbations of the heavenly bodies, four hours.—By DR. F. LONDON: Definite integrals, three hours; Mathematics of insurance, two hours.

UNIVERSITY OF ERLANGEN.—By Professor P. GORDAN: Theory of differential equations, four hours; Theory of

invariants, four hours; Seminar, three hours.—By Professor M. NOETHER; Differential and integral calculus, II, four hours; Descriptive geometry, with exercises, two hours; Selected chapters in mechanics, two hours.

UNIVERSITY OF FREIBURG.—By Professor J. LÜROTH: Theory of functions, five hours; Theoretical astronomy, two hours.—By Professor L. STICKELBERGER: Integral calculus, four hours; Higher plane curves, three hours; Seminar.—By Dr. A. LOEWY: Algebraic equations, four hours; Foundations of geometry, two hours.—By Dr. E. REBMANN: Elements of projective geometry, two hours.

UNIVERSITY OF GIESSEN.—By Professor M. PASCH: Foundations of geometry, four hours; Elements of algebra, two hours; Mathematical seminar, one hour.—By Professor E. NETTO: Plane analytic geometry, four hours; Introduction to the theory of functions, two hours; Seminar, one hour.—By Professor R. HAUSSNER: Differential geometry, with exercises, five hours; Technical mechanics, second part, with exercises, four and one half hours.

UNIVERSITY OF GREIFSWALD.—By Professor W. THOMÉ: Mechanics II, four hours; Theory of potential, four hours; Seminar, two hours.—By Professor E. STUDY: Analytic geometry I, four hours; Introduction to synthetic geomtry, two hours; Select chapters in the theory of functions, one hour; Seminar, one hour.

UNIVERSITY OF HEIDELBERG.—By Professor L. KOENIGSBERGER: Infinitesimal calculus, four hours; Theory of functions, four hours; Seminar, two hours.—By Professor M. CANTOR: Algebraic analysis, four hours; Arithmetic and algebra, three hours.—By Professor K. KOEHLER, Analytic geometry, three hours.—By Professor G. LANDSBERG: Theory of determinants, two hours; Algebraic curves, two hours.—By Dr. K. BOEHM: Introduction to the theory of continuous transformation groups, with applications to geometry and differential equations, one or two hours; Gauss's memoir on general theory of forces attracting or repelling as the inverse square of distance, one hour.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Differential equations, four hours; Plane analytic geometry, four hours.—By Professor A. GUTZMER: Differential calculus, five hours; Theory of potential, four hours; Seminar.—By Professor G. FREGE: The foundations of arithmetic, four hours; Mathematical evercises, two hours.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER: Introduction to algebra and determinants, four hours; Theory of elliptic functions, four hours; Seminar, one hour. —By Professor P. STÄCKEL: Infinitesimal calculus and introduction to analysis, four hours; Advanced dynamics of rigid bodies, four hours; History of mathematics since the Renaissance, one hour; Seminar, one hour.

UNIVERSITY OF KOENIGSBERG.—By Professor F. MEYER: Introduction to the theory of numbers, four hours; Seminar, one hour.—By Professor A. SCHOENFLIES: Differential geometry of curves and surfaces, four hours; Seminar, one hour.—By Professor L. SAALSCHÜTZ: Differential calculus, four hours, with seminar one hour; Gauss's trigonometric and other interesting series, two hours.—By Dr. J. RAHTS: Higher geodesy, two hours; Least squares, two hours.—By Dr. T. VAHLEN: Analytic geometry of the plane and of space, four hours, with exercises, one hour.—By Dr. E. MÜLLER: Introduction to descriptive geometry, two hours.

UNIVERSITY OF LEIPSIC: By Professor C. NEUMANN: Theory of potential and spherical harmonics, four hours; Seminar, two hours.—Professor A. MAYER: Courses to be announced later.—By Professor O. HÖLDER: Ordinary differential equations, four hours; Scientifically rigorous foundation of arithmetic, one hour; Seminar, one hour.— By Professor F. ENGEL: Analytic geometry of the plane and of space, four hours; Differential invariants (continued), one hour; Projective geometry and theory of invariants (continuation of projective geometry), two hours; Seminar, with Dr. G. Kowalewski, one hour.—By Dr. F. HAUSDORFF: Differential geometry, four hours; Mengenlehre, two hours.—By Dr. G. KOWALEWSKI; Algebraic analysis, four hours, with exercises, one hour; Line geometry, two hours; Seminar on differential invariants with Professor Engel, one hour.—By Dr. H. LIEBMANN: Algebraic equations, two hours; Synthetic geometry of conics, two hours; Selected chapters in the theory of numbers, one hour.

UNIVERSITY OF MARBURG.—By Professor F. SCHOTTKY: Infinitesimal calculus, four hours; Calculus of variations, two hours; Seminar, four hours.—By Professor E. HESS: Analytic and synthetic treatment of plane geometry, four hours; Integration of differential equations, three hours; Selected chapters on applied mechanics, two hours; Seminar, four hours.—By Dr. F. v. DALWIGK: Introduction to

mathematics for chemists, three hours, with exercises, one hour; Collineation and perspective, two hours.

UNIVERSITY OF STRASSBURG.—By Professor T. REYE: Selected chapters on higher synthetic geometry, three hours; Theory of forces which attract according to Newton's law, (theory of potential), three hours; Seminar, two hours.—By Professor H. WEBER: Theory of abelian functions, four hours; Applications of elliptic functions to mechanics and physics, two hours; Seminar, two hours.—By Professor G. ROTH: Differential and integral calculus, three hours, with exercises, two hours; Plane analytic geometry, three hours.—By Professor A. KRAZER: Definite integrals, three hours; Analytic geometry of space, three hours; Theory of the theta functions, two hours; Seminar, one hour.—By Dr. E. TIMERDING: Theory of surfaces, two hours.—By Dr. J. WELLSTEIN: Descriptive geometry, II, two hours.

UNIVERSITY OF TÜBINGEN.—By Professor A. v. BRILL: Analytic geometry of space, three hours; Theory of curvature of surfaces, four hours; Seminar, in two sections, two hours.—By Professor H. STAHL: Elementary analysis, three hours, with exercises, one hour; Higher analysis, I (differential calculus), four hours, with exercises, one hour.—By Professor L. MAURER: Synthetic geometry, two hours, with exercises, one hour; Descriptive geometry, I, two hours, with exercises, two hours.

UNIVERSITY OF WÜRZBURG.—By Professor F. PRYM: Integral calculus, six hours, with exercises, two hours; Seminar on topics in the theory of functions, two hours.—By Professor A. VOSS: Analytic and synthetic geometry of conics, four hours; Analytic mechanics, four hours; Seminar on topics in higher mathematics, two hours.—By Professor E. SELLING: Partial differential equations of mathematical physics, four hours; Theory of planetary motions, three hours.

UNIVERSITY OF ZURICH.—By Professor H. BURKHARDT: Algebraic analysis, four hours; Partial differential equations, four hours; Seminar, two hours.—By Professor A. WEILER: Analytic geometry, II, two hours; Descriptive geometry, II, two hours; Map projection, two hours; Synthetic geometry, II, two hours.

THE dedication of the monument to FRANCESCO BRIOSCHI took place on December 13, 1900, the third anniversary of

his death. The monument, which is of bronze, has been placed in the Royal Technical Institute of Milan. It is the work of the sculptor L. Secchi, of Milan, and was erected with a part of the money collected by subscription from the friends, admirers, and pupils of BRIOSCHI. At the inauguration addresses were made by Professors G. COLOMBO, P. BLASERNA, G. CELORIA, and G. BARDELLI. These addresses are published in the *Annali di Matematica* for January, 1901 (3d series, volume V., page 141).

OWING to the state of his health, Professor P. G. TAIT of the chair of natural philosophy in the University of Edinburgh, has intimated his approaching resignation, after a distinguished service of over forty years.

DR. J. W. L. GLAISHER has been elected president of the Royal astronomical society of Great Britain for the forthcoming year. The honorary secretaries are Messrs. F. W. DYSON and E. T. WHITTAKER, and the foreign secretary is, as before, Sir WILLIAM HUGGINS. The gold medal of the society was this year awarded to Professor E. C. PICKERING, of Harvard University.

PROFESSOR A. PRINGSHEIM has been promoted to a full professorship in the University of Munich.

PROFESSOR M. CANTOR has been chosen corresponding member of the Royal institute of science, letters, and art of Venice.

IN justice to Professor C. N. LITTLE, whose resignation from the faculty of the Leland Stanford Jr. University was noted in the preceding number of the BULLETIN, it is proper to state that Professor Little's resignation was tendered voluntarily as an expression of disapproval of the administrative policy of the university.

PROFESSOR D. E. SMITH, principal of the Brockport, N. Y., State normal school, has accepted a call to the professorship of mathematics in Teachers College, Columbia University, and will enter upon the duties of his new position at the beginning of the next academic year.

MISS E. F. PENDLETON, instructor in mathematics at Wellesley College, has been elected dean of the college. Miss HELEN MERRILL has been promoted to an associate professorship of mathematics. Miss R. H. VIVIAN, alumnæ fellow in mathematics at the University of Pennsylvania, has been appointed instructor in mathematics at Wellesley.

DR. G. A. MILLER, of Cornell University, has been appointed to an assistant professorship of mathematics at Leland Stanford University.

DR. O. SCHLÖMILCH, formerly professor of mathematics at the Technical School at Dresden, died on February 7th, at the age of seventy-eight years.

PROFESSOR CHARLES MCDONALD, died at Halifax, N. S., on March 10th, after a service of over thirty years in Dalhousie College.

DR. J. M. RICE died at Northboro, Mass., on March 2d, aged sixty-eight years. He was appointed professor of mathematics at the Naval academy at Annapolis in 1870.

PROFESSOR G. P. STARKWEATHER, of Yale University, died on March 21st, at the age of twenty-eight years.

THE library of the late Professor E. B. CHRISTOFFEL is offered for sale by Gustav Fock of Leipzig.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BELTRAMI (L.). Francesco Brioschi: nel giorno della morte; un anno dopo; davanti al monumento, dicembre 1900. Milano, Allegretti, 1900. 8vo. 36 pp., portrait.

BÔCHER (M.). The theory of linear dependence. Salem, Mass., Salem Press, 1901. 4to. 16 pp. (From the *Annals of Mathematics*, second series, Vol. 2.)

BORINI (B.). I continuanti. Forli, Medri, 1900. 4to. 117 pp.

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